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Paper No. 872
RADIO SECTION

DELAY NETWORKS HAVING MAXIMALLY FLAT FREQUENCY CHARACTERISTICS

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(*The paper was first received 14th March, and in revised form 4th July*, 1949.)

SUMMARY

A lumped-constant equivalent of a transmission line can be obtained in general in the form of a symmetrical lattice, in which the series and lattice arms are inverse and approximate respectively to the short-circuit and open-circuit impedances of half the line. One such set of approximations can be derived from the infinite ladder networks (Cauer's canonical form) equivalent to these impedances.

These approximations produce all-pass constant-impedance networks (dissipation being neglected) in which the delay is maximally flat in the sense that the first $2m - 1$ derivatives of the delay with respect to frequency are zero at the origin; m is an integer expressing the order of the approximation.

LIST OF SYMBOLS

m = Order of approximation.
R = Impedance of line or network, ohms.
ω = Angular frequency, radians/sec.
T = Half delay of line or network, sec.
$x = \omega T$.
X_m = Normalized reactance.
B = Phase-shift, radians.
$A_{m,n}$ = Numerical coefficient.
$R_{m,\nu}$ = Lommel polynomial.
Γ = Gamma function.

(1) INTRODUCTION

A delay line is a system for which the output is a replica of the input in shape, but is delayed in time by some specified amount. Such characteristics can, strictly speaking, be obtained only from a transmission line, i.e. a system with distributed constants. For many practical purposes, however, a more compact and sufficiently good unit can be made in the form of a delay network, or artificial line, in which the components are lumped constants. One of the limitations of such a network is that its frequency characteristics can approximate to those of a true delay line only over a finite band of frequencies. The particular band chosen depends on the practical application, but a widely used band is that extending from zero up to some specified frequency, and it is this band which is here dealt with. The problem, then, is to design a network in which the loss is zero and the phase-shift proportional to frequency, over this band.

There have been two distinct approaches to this problem. The first leads to a low-pass filter in the form of a ladder network, the pass band of the filter corresponding, more or less, to the specified frequency band. In the second, that with which we are concerned, the network is designed as a single lattice, although it may be broken up into simpler lattices or converted into unbalanced networks for construction.

(2) GENERAL

The problem of finding a lattice whose characteristics approximate to those of a transmission line has been discussed by Guillemin[1]; he shows that an exact equivalent of a dissipationless line of impedance R and delay $2T$ has series arms and lattice arms whose reactances are $R \tan \omega T$ and $- R \cot \omega T$ respectively.* Networks which have these reactance functions can be determined by the methods of Foster and Cauer; the resultant networks for $\tan \omega T$ are shown in Fig. 1. Since $- R \cot \omega T$ is

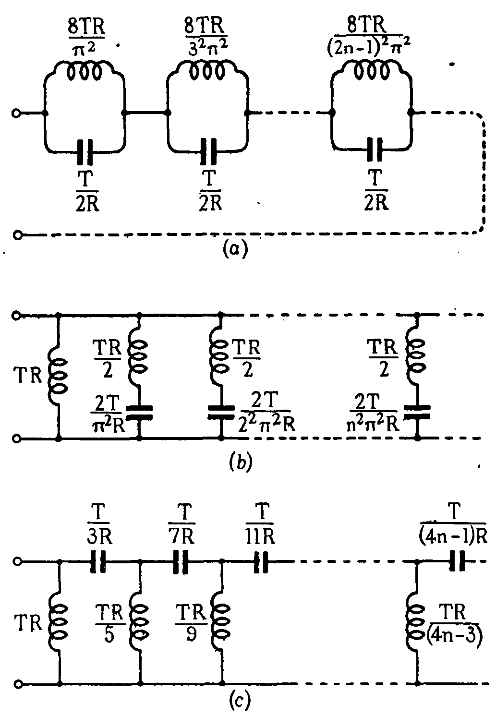

Fig. 1.—Networks equivalent to short-circuit impedance of transmission line: impedance, R; delay, T; reactance, $R \tan \omega T$.

the inverse of $R \tan \omega T$ with respect to R, it may be represented by networks obtained by reciprocation in the usual way.

These networks are infinite in extent; the approximately equivalent delay network must have a finite number of components. If RX is the reactance of the series arm of a symmetrical lattice, it will, in general, be possible to find the inverse network, of reactance $- R/X$, to form the lattice arms. The characteristics of such a lattice are:

(*a*) *Impedance: R*
(*b*) *Insertion characteristics (between terminations R):*

loss: zero

phase-shift: $B = 2 \arctan X$

envelope delay: $\dfrac{dB}{d\omega} = \dfrac{2}{1 + X^2}\dfrac{dX}{d\omega}$

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* The total delay of such a lattice can be split into two equal parts; one associated with the series arm and one with the lattice arm. It is thus more convenient to use half the total delay as the unit.

For the delay network, X must be chosen so that the delay is reasonably constant from zero to the specified upper frequency. Delay can be specified as phase delay B/ω or envelope delay $dB/d\omega$; for the present analysis, envelope delay is more convenient.

Previous methods[1,2] of choosing X have used the fact that X can be specified in terms of its zeros and poles, these being chosen to be identical with some of the zeros and poles of $\tan \omega T$, which occur at

$$\omega = n\pi/2T \begin{cases} \text{zeros: } n = 0, 2, 4, \ldots \\ \text{poles: } n = 1, 3, 5, \ldots \end{cases}$$

If only those poles and zeros within the required band are chosen, the resultant delay oscillates about its wanted value, the oscillations increasing in amplitude towards the edge of the band. These oscillations may be removed by means of additional poles and zeros outside the band, not coinciding with those of $\tan \omega T$, but methods of choosing them depend, to some extent, on trial and error.

It will now be shown that the network of Fig. 1(*c*) can be used to form a series of delay networks in which the delay characteristics are of the type known as "maximally flat." It is immaterial whether envelope or phase delay is used: if one is maximally flat, so is the other. The delay of any physical network may be expressed as a Maclaurin series:

$$\text{Delay} = T_0(1 + a_2\omega^2 + a_4\omega^4 + \ldots)$$

where T_0 is the zero-frequency delay and the a's are constants. To achieve maximal flatness with r available design parameters, these must be chosen so that

$$a_2 = a_4 = \ldots = a_{2r} = 0$$

Another way of expressing this is to say that the first $2r+1$ derivatives of the delay with respect to frequency are zero at the origin. The greater r is, the greater will be the range over which the delay is reasonably constant.

(3) MAXIMALLY FLAT DELAY NETWORKS

Network (*c*) of Fig. 1 may be approximated by breaking the network off after as many components as desired; let the number of components retained be m. If the last component retained is a capacitor, the remainder of the network is replaced by a short-circuit; if an inductor, an open-circuit is used. The first three such approximations are shown in Fig. 2(*a*); Fig. 2(*b*) gives the corresponding lattices, the series arms of which are the reactances of (*a*); each lattice arm is the inverse with respect to R of the appropriate series arm. These lattices are thus constant-impedance all-pass networks of the type discussed in Section 2; their phase characteristics depend on the reactance RX_m of the series arm. Expressions for X_m for m up to 6 are given in Table 1; in this Table and subsequently $x = \omega T$.

The Table may be extended by recurrence formulae: if N_m and D_m are the numerator and denominator of the mth order fraction then

$$N_m = (2m-1)N_{m-1} - N_{m-2}x^2$$

$$D_m = (2m-1)D_{m-1} - D_{m-2}x^2$$

the initial values being

$$N_0 = 0,\ N_1 = x;\ D_0 = 1,\ D_1 = 1.$$

It is of interest to determine the critical frequencies (zeros and poles) of these reactances and to compare them with the critical frequencies of $\tan x$. The results are shown in Table 2, in which each entry is the ratio of the critical frequency of X_m to the corresponding critical frequency of $\tan x$.

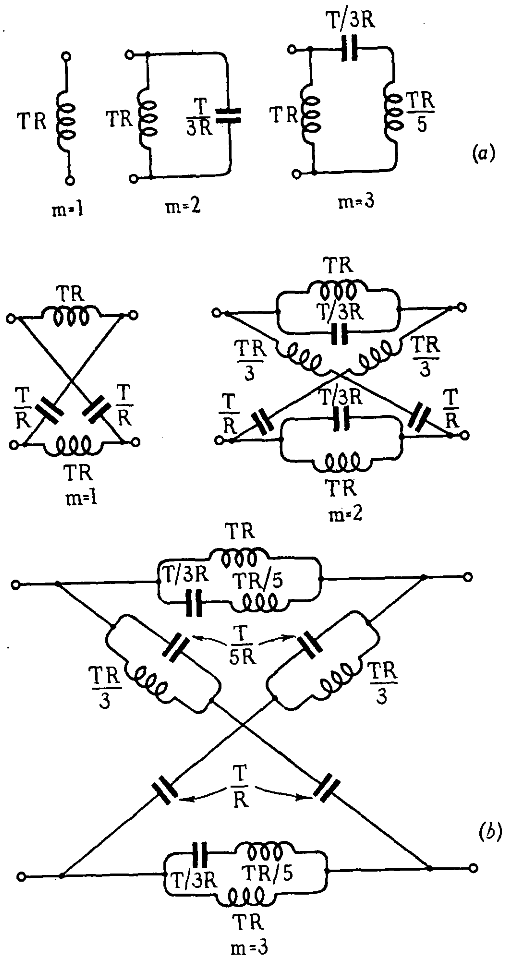

Fig. 2.—(*a*) Successive approximations to network (c) of Fig. 1.
(*b*) Lattices using networks of (*a*).

Table 1

Order of approximation m	Normalized reactance
1	x
2	$\dfrac{3x}{3 - x^2}$
3	$\dfrac{x(15 - x^2)}{15 - 6x^2}$
4	$\dfrac{x(105 - 10x^2)}{105 - 45x^2 + x^4}$
5	$\dfrac{x(945 - 105x^2 + x^4)}{945 - 420x^2 + 15x^2}$
6	$\dfrac{x(10\,395 - 1\,260x^2 + 21x^4)}{10\,395 - 4\,725x^2 + 210x^4 - x^6}$

The first critical frequency, zero, is the same for $\tan x$ and all the approximations, but for the others the critical frequencies of the approximations are higher; for the lower critical frequencies of higher order the difference is negligible. The situa-

Table 2

Order m	Corresponding critical frequencies of tan x						
	0	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π
1	1·00	∞	—	—	—	—	—
2	1·00	1·10	∞	—	—	—	—
3	1·00	1·01	1·23	∞	—	—	—
4	1·00	1·00	1·03	1·38	∞	—	—
5	1·00	1·00	1·00	1·07	1·55	∞	—
6	1·00	1·00	1·00	1·02	1·13	1·82	∞

tion is thus rather similar to that resulting from existing methods: for the 6th order, for example, there are three coincident critical frequencies, corresponding roughly to the band within which the delay is constant, and three external non-coincident frequencies.

The numerator is equal to the denominator with the last term omitted.

For all orders of approximation the envelope delay equals $2T$ multiplied by a fraction which is a function of x; the zero-frequency value of this fraction is unity and as x ($= \omega T$) increases, the value of this fraction drops monotonically to zero, since numerator and denominator are always positive and the denominator always exceeds the numerator by x^{2n}.

The fact that the coefficients of x^0 to x^{2m-2} are equal in numerator and denominator means that the delay has maximal flatness as defined in Section 2; the constant r of that Section is equal to $m - 1$. In effect, of the m design parameters available, one is used to get the correct delay and the remainder are used to give maximal fitness.

Curves showing the relative envelope delay for various values of m are given in Fig. 3; the corresponding curves for phase delay are similar in shape but do not fall off quite so rapidly. For a given value of delay ($= 2T$) and upper frequency limit, the corresponding value of x ($= \omega T$) can be calculated; the curves will then show the value of m required to maintain the delay constant up to this limit.

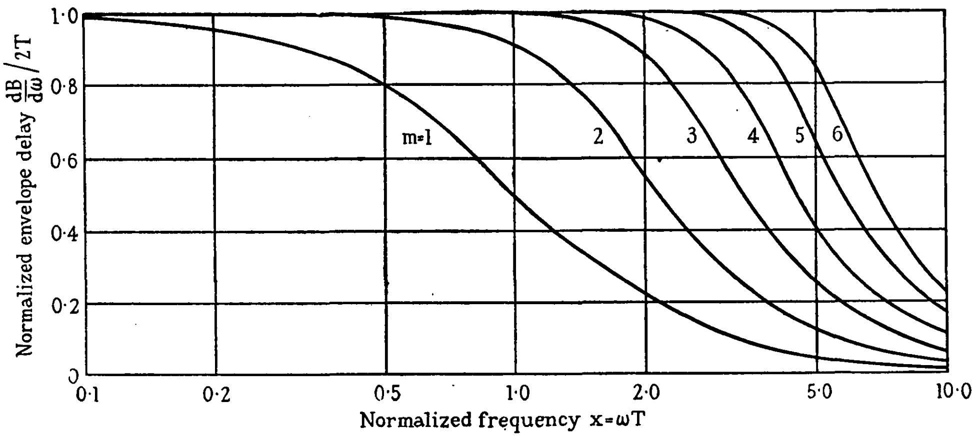

Fig. 3.—Envelope delay of lattice networks.

The envelope delay of these lattices is given by

$$\frac{2}{1 + X_m^2}\frac{dX_m}{d\omega}$$

Values of this expression, corresponding to the values of X_m in Table 1, are given in Table 3, as far as $m = 4$. These expressions may be derived from recurrence formulae; if $A_{m,n}$ is the coefficient of x^{2n} in the denominator of the mth order fraction, then

Table 3

Order m	Envelope delay
1	$\frac{1}{1 + x^2}2T$
2	$\frac{9 + 3x^2}{9 + 3x^2 + x^4}2T$
3	$\frac{225 + 45x^2 + 6x^4}{225 + 45x^2 + 6x^4 + x^6}2T$
4	$\frac{11\,025 + 1\,575x^2 + 135x^4 + 10x^6}{11\,025 + 1\,575x^2 + 135x^4 + 10x^6 + x^8}2T$

$$A_{m,n} = (2m/n - 1)A_{m-1,n-1} \quad n \neq 0$$

$$A_{m,0} = (2m - 1)^2 A_{m-1,0}$$

with the starting point

$$A_{0,0} = 1.$$

(4) REFERENCES

(1) GUILLEMIN, E. A.: "Communication Networks," Vol. II (John Wiley, 1935).

(2) WALD, M.: "Zur Theorie des Phasenausgleichs," *Elektrische Nachrichten-Technik*, 1942, **19**, p. 1.

(3) WATSON, G. N.: "Theory of Bessel Functions" (Cambridge University Press, 1944).

(5) APPENDIX

The proofs of the properties set out in the body of the paper are given here. They depend on various theorems on Lommel polynomials, which are discussed by Watson[3]; the numbers in square brackets refer to paragraphs in his book.

Network (*c*) of Fig. 1 is associated with the continued-fraction expansion

$$\tan x = \frac{1}{1/x -}\,\frac{1}{3/x -}\,\frac{1}{5/x -}\cdots\frac{1}{(2m-1)/x -}\cdots$$

and the successive approximations shown in Fig. 2 are R times the successive convergents of this fraction. The mth convergent may be expressed as a ratio of Lommel polynomials [9.65]; the reactances X_m of Table 1 are given by

$$X_m = \frac{R_{m-1,\frac{3}{2}}(x)}{R_{m,\frac{1}{2}}(x)}$$

where $R_{m,\nu}(x)$ is a Lommel polynomial defined by [9.61]

$$R_{m,\nu}(x) = \sum_{n=0}^{\leq \frac{1}{2}m} \frac{(-)^n (m-n)!\,\Gamma(\nu+m-n)2^{m-2n}}{n!(m-2n)!\,\Gamma(\nu+n)x^{m-2n}}$$

Since all polynomials here have x as argument, the x will be omitted subsequently. The Lommel polynomials, it will be seen, are polynomials in $1/x$, and the expressions of Table 1 have been converted to ratios of polynomials in x by multiplying above and below by x^m.

From the general recurrence formula [9.63]

$$R_{m-1,\nu} + R_{m+1,\nu} = [2(\nu+m)/x]R_{m,\nu}$$

it can be deduced that

$$R_{m,\frac{1}{2}} = [(2m-1)/x]R_{m-1,\frac{1}{2}} - R_{m-2,\frac{1}{2}}$$
$$R_{m-1,\frac{3}{2}} = [(2m-1)/x]R_{m-2,\frac{3}{2}} - R_{m-3,\frac{3}{2}}$$

from which the recurrence formulae associated with Table 1 are derived.

The envelope delay obtained by using the reactance X_m in a lattice is

$$\frac{2}{1+X_m^2}\frac{dX_m}{d\omega} = 2\frac{dx}{d\omega}\frac{X_m'}{1+X_m^2}$$

where the prime denotes differentiation with respect to x. The factor $2dx/d\omega$ is the factor $2T$ of Table 3; $X_m'/(1+X_m^2)$ is the ratio of polynomials in x. In terms of Lommel polynomials,

$$\frac{X_m'}{1+X_m^2} = \frac{R_{m,\frac{1}{2}}R'_{m-1,\frac{3}{2}} - R'_{m,\frac{1}{2}}R_{m-1,\frac{3}{2}}}{R^2_{m-1,\frac{3}{2}} + R^2_{m,\frac{1}{2}}}$$

By means of the general relations [9.63]

$$R'_{m,\nu} = [(m+2)/x]R_{m,\nu} + R_{m+1,\nu-1} - R_{m+1,\nu}$$
$$R'_{m,\nu} = [(2\nu+m)/x]R_{m,\nu} - R_{m-1,\nu+1} - R_{m+1,\nu}$$

the numerator of this fraction can be expressed as

$$\begin{aligned} &R_{m,\frac{1}{2}}\{[(m+1)/x]R_{m-1,\frac{3}{2}} + R_{m,\frac{1}{2}} - R_{m,\frac{3}{2}}\} \\ &\qquad - R_{m-1,\frac{3}{2}}\{[(m+1)/x]R_{m,\frac{1}{2}} - R_{m-1,\frac{3}{2}} - R_{m+1,\frac{1}{2}}\} \\ &= R^2_{m,\frac{1}{2}} + R^2_{m-1,\frac{3}{2}} - (R_{m,\frac{1}{2}}R_{m,\frac{3}{2}} - R_{m-1,\frac{3}{2}}R_{m+1,\frac{1}{2}}) \\ &= R^2_{m,\frac{1}{2}} + R^2_{m-1,\frac{3}{2}} - 1 \end{aligned}$$

since

$$R_{m,\nu}R_{m,\nu+1} - R_{m+1,\nu}R_{m-1,\nu+1} = 1 \quad [9.64]$$

Hence, using

$$R^2_{m,\frac{1}{2}} + R^2_{m-1,\frac{3}{2}} = (-)^m R_{2m,\frac{1}{2}-m} \quad [9.62]$$

it can be deduced that

$$\frac{X_m'}{1+X_m^2} = \frac{(-)^m R_{2m,\frac{1}{2}-m} - 1}{(-)^m R_{2m,\frac{1}{2}-m}}$$

from which the expressions of Table 3 may be derived; multiplication above and below by x^{2m} is necessary to give polynomials in x.

From the general formula for $R_{m,\nu}$ quoted earlier it may be shown that

$$(-)^m R_{2m,\frac{1}{2}-m} = \sum_{n=0}^{m} \frac{(2m-n)!(2m-2n)!}{2^{2m-2n}n![(m-n)!]^2}\frac{1}{x^{2m-2n}}$$

from which the recurrence formulae associated with Table 3 may be deduced.